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A COMPUTER MODEL OF HOLE-PRESSURE MEASUREMENT IN POISEUILLE FLOW OF VISCO-ELASTIC LIQUIDS

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A COMPUTER MODEL OF HOLE-PRESSURE MEASUREMENT IN POISEUILLE FLOW OF VISCO-ELASTIC LIQUIDS

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ABSTRACT

Liquid-filled holes used for pressure measurements of visco-elastic liquids give rise to systematic hole pressure 'errors'.[‡] Tanner and Pipkin have presented analysis for flows of a second order fluid in which they derive a simple relation between the first normal stress difference and the hole pressure for flow situations where Reynolds numbers are very small. Implicit in the analysis is the assumption that the streamlines are symmetric about the hole center line. In this paper, using a numerical solution, we investigate the relationship between the hole pressure and the first normal stress difference for a range of Reynolds numbers where inertial effects are not negligible. The ratio of hole pressure/first normal stress difference is found to vary from 0.25 to 0.16 as the Reynolds number is varied from 1 to 10. Streamline patterns are presented for Poiseuille flow of a second order fluid over a slot cut into one wall of an otherwise straight channel. Various geometries are considered. The results naturally include those for an incompressible Newtonian liquid at non-zero Reynolds numbers.

AMS(MOS) Subject Classification 70.65 Work Unit No. 3 - Applications of Mathematics
Key Words: Visco-elastic liquids, Hole pressure measurements, Numerical Methods

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[‡] It has been customary in the past to apply the term hole pressure 'error' to the quantity $P_H - P_W$ where P_H is the pressure measured in the static liquid at the end of a hole in a channel wall and P_W is the pressure which would be exerted on the wall by liquid flowing in the channel if the hole were not present. The basic aim of this paper is to relate this 'quantity' to material properties of the liquid, and as such the term 'error' is misleading since the 'quantity' will itself be subject to error. We choose, therefore, to refer to the 'hole pressure' defined as $P_W - P_H$. It will be noted that the sign of the hole pressure is chosen to be positive for elastic liquids.

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Significance and Explanation

For some time now the polymer processing industry has put considerable effort into the design of simple, reliable instruments for measuring material properties such as viscosity, elasticity, etc, of polymeric liquids or polymer melts.

This report considers the theoretical background to one such instrument which gives elasticity readings by measurement of pressure values in a liquid when it flows through a straight channel. One wall of the channel has a slot cut into it, and pressure is measured at the bottom of the slot and at a point on the channel wall immediately opposite the slot. The report looks in detail at the effect of inertial contributions to the flow and in particular shows how inertial distortion modifies the relationship between the elasticity of the liquid and the measured pressure values. Streamline patterns are presented showing how much distortion one can expect for Reynolds numbers up to 25.

The responsibility for the wording and views expressed in this descriptive summary lies with MRC, and not with the author of this report.

A COMPUTER MODEL OF HOLE-PRESSURE MEASUREMENT IN POISEUILLE FLOW OF VISCO-ELASTIC LIQUIDS

P. Townsend

Introduction

The search for simple reliable instruments which measure material properties of visco-elastic liquids is a never-ending one. In this paper we use a numerical model to investigate one flow situation in which the use of liquid-filled holes or slots for pressure measurements of flowing visco-elastic liquids gives rise to a direct measurement of the elasticity of the liquid.

In a number of very detailed and careful experiments carried out by Lodge and his co-workers [1, 2 and 3] it became clear that errors are produced if pressure measurements of visco-elastic liquids are taken from transducers mounted at the bottom of liquid-filled holes. Some work with a polyisobutylene solution indicated an approximately linear relationship between the error in the pressure reading and the first normal stress difference. Lodge recognized that due to the apparently systematic relationship between the 'hole pressure' and the liquid elasticity, a basis exists for an instrument which would give direct readings of elastic properties of a liquid from a simple shear flow. These ideas are now embodied in the Seiscor-Lodge Stressmeter [4].

A simple analysis due to Tanner and Pipkin [5] for Poiseuille flow of a second order liquid over a slot showed that for very small Reynolds numbers where inertial effects were negligible, the hole pressure was precisely one quarter of the first normal stress difference. This result was later confirmed by Higashitani and Pritchard [6] using a slightly different approach. Kearsley [7] derived a similar result relating hole pressure and the second normal stress difference for slots placed along the main flow direction. The restrictions placed on these theoretical analyses were quite severe, however, and the very simple results obtained unlikely to be valid when fluid inertia is non-negligible. The investigation of the effects of inertia on the relationship between the hole pressure and the first normal stress difference in a visco-elastic liquid is the aim of this paper.

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In order to mathematically model a visco-elastic liquid, one is faced with the very difficult question of which constitutive equations to use. Certainly one would like to be able to model as closely as possible as many fluid properties as one can, but often the more sophisticated fluid models give rise to partial differential equations which are either too difficult or too expensive to solve. In the flow situations we shall consider, certainly our first concern is for fluid elasticity effects, and one of the simplest models available to us is the second order fluid. One has to be careful in interpreting results obtained in this case for, as Pipkin [8] points out, the results are valid only if the presence of second-order terms causes a small perturbation of Newtonian flow. For flows which differ considerably from the Newtonian situation one may argue that a second-order analysis gives one only preliminary insight into how Newtonian flow will be affected by slight elasticity. In the present analysis it proves to be the case that the elastic solutions obtained, except for pressure considerations, differ very little from Newtonian flow.

The equations governing the flow problem, even for the Newtonian case, are too difficult for an analytical solution, and we must turn therefore to a numerical model. Some of the earliest work done in this area was that by Thom and Apelt [9] who derived a perturbation solution for small Reynolds numbers for the flow of a Newtonian liquid. They attempted to write down a simple relationship between the hole pressure and the Reynolds number. Some comparison of their results is made with experimental work due to Ray [10]. O'Brien [11, 12 and 13] has calculated Stokes flow of a Newtonian liquid past slots of various geometries and in situations where the unperturbed flow in the main channel is taken to be both Couette flow and Poiseuille flow and also some combination of the two. Results are presented indicating the dependence of streamline patterns on depth-to-length slot ratios but unfortunately no reference is made to hole pressure measurements.

A numerical solution of the full Navier-Stokes equations for Newtonian Poiseuille flow past a slot is presented by Schefenacker [14], although it is not clear why inertia terms are retained for the values of the Reynolds number considered are extremely small. The geometry used in this work is derived from extension applications and is not directly relevant here. Numerical solutions for high Reynolds number Newtonian flow have been obtained by Stevenson [15], who considers the flow in a tube which has a circumferential wall cavity and calculates the streamlines. Again, however, there is no reference to the hole pressure.

Hole-pressure calculations for a second-order fluid have been made by Malkus [16], who presents results for the ratio of hole-pressure to first normal stress difference as a function of depth-to-width ratio of the slot. Both Couette and Poiseuille flow are considered but the analysis is limited to Stokes flow.

In a recent paper by Crochet and Bezy [17], consideration is given to the flow of a Maxwell-type liquid in a geometry similar to that adopted in this paper. Only a limited number of results are presented since the problem is essentially only solved as one test of a new finite element technique. The authors express some doubt about certain features of their solution due to numerical difficulties which they experienced. We shall make further reference to this work later.

Theoretical Analysis

In this analysis we take a rectangular Cartesian coordinate system (x,y,z) and assume that we have steady two-dimensional flow between two infinite flat plates AH and BG , which are positioned parallel to the OXZ plane at a distance h apart (see Figure 1). The lower plate is assumed to have cut into it a rectangular slot CDEF parallel to the Z axis, of depth d and width b , and of infinite length.

If we take a velocity vector $\underline{v} \equiv (u,v,0)$ then the equations which govern the notion of the liquid are

$$\rho\left(u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} \right) = - \frac{\partial P}{\partial x} + \frac{\partial P'_{xx}}{\partial x} \quad \frac{\partial P'_{xy}}{\partial y} , \tag{1}$$

$$\rho\left(u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} \right) = - \frac{\partial P}{\partial y} + \frac{\partial P'_{xy}}{\partial x} + \frac{\partial P'_{yy}}{\partial y} , \tag{2}$$

and

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{3}$$

where ρ is the density of the liquid, P denotes the isotropic pressure and P'_{xx} , P'_{xy} and P'_{yy} the relevant components of the extra stress tensor. The boundary conditions which apply are that the velocity components are zero on all plate surfaces, and that provided one is sufficiently far from the slot then the flow is undisturbed Poiseuille flow. To complete the specification of the problem we need to choose appropriate equations of state for the liquid. For reasons given in the introduction we confine attention to the Rivlin-Ericksen incompressible second-order fluid given by

$$P'_{ik} = 2\alpha_1 e^{(1)}_{ik} + 2\alpha_2 e^{(2)}_{ik} + 4\alpha_3 e^{(1)}_{im} e^{(1)m}_{k} \tag{4}$$

where α_1, α_2 and α_3 are material constants and $e^{(j)}_{ik}$ is the jth rate of strain tensor introducted by Oldroyd [18]†. This fluid exhibits both first and second normal stress differences, governed by non-zero values of the parameters

† $e^{(j)}_{ik} = (1/2)A^{(j)}_{ik}$ where $A^{(j)}_{ik}$ is the jth Rivlin - Ericksen tensor.

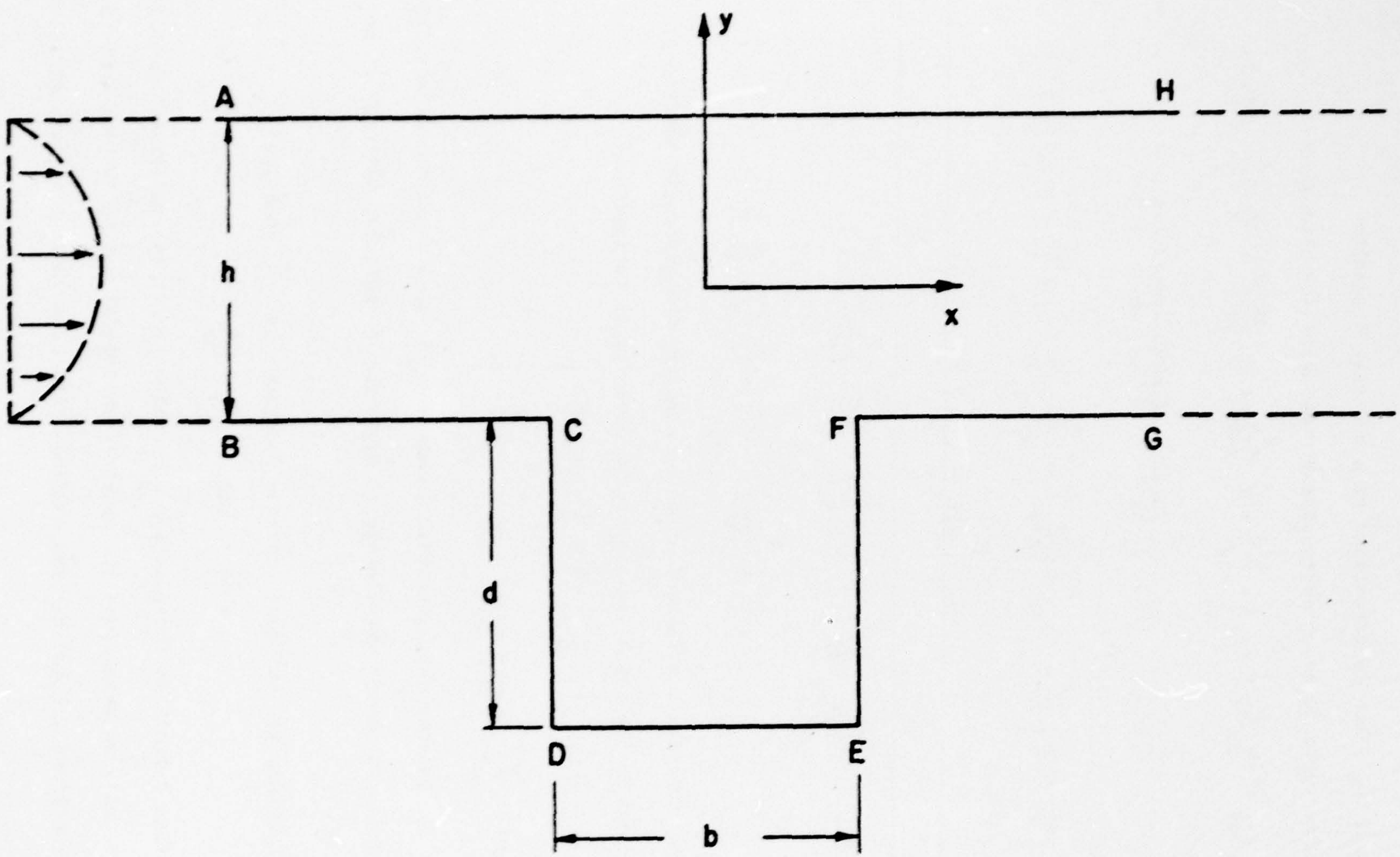

Figure 1
The channel/slot geometry

α_2 and α_3 , but is restricted to a constant viscosity α_1 .

The three relevant extra stress components are then given by

$$
\begin{aligned}
P'_{xx} = 2\alpha_1 \frac{\partial u}{\partial x} + 2\alpha_2 \left[u \frac{\partial^2 u}{\partial x^2} + v \frac{\partial^2 u}{\partial x \partial y} + 2 \left(\frac{\partial u}{\partial x} \right)^2 + \frac{\partial v}{\partial x} \left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \right) \right] \\
+ 4\alpha_3 \left[\left(\frac{\partial u}{\partial x} \right)^2 + \frac{1}{4} \left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \right)^2 \right] ,
\end{aligned}
\tag{5}
$$

$$
\begin{aligned}
P'_{yy} = 2\alpha_1 \frac{\partial v}{\partial y} + 2\alpha_2 \left[u \frac{\partial^2 v}{\partial x \partial y} + v \frac{\partial^2 v}{\partial y^2} + \frac{\partial u}{\partial y} \left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \right) + 2 \left(\frac{\partial v}{\partial y} \right)^2 \right] \\
+ 4\alpha_3 \left[\left(\frac{\partial v}{\partial y} \right)^2 + \frac{1}{4} \left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \right)^2 \right] ,
\end{aligned}
\tag{6}
$$

and

$$
\begin{aligned}
P'_{xy} = \alpha_1 \left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \right) + 2\alpha_2 \left[\frac{u}{2} \left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 u}{\partial x \partial y} \right) \right. \\
\left. + \frac{v}{2} \left(\frac{\partial^2 v}{\partial x \partial y} + \frac{\partial^2 u}{\partial y^2} \right) + \frac{\partial v}{\partial x} \frac{\partial v}{\partial y} + \frac{\partial u}{\partial x} \frac{\partial u}{\partial y} \right] .
\end{aligned}
\tag{7}
$$

We now take a number of steps to simplify the governing equations. First we introduce the following set of non-dimensional variables

$$
\left.
\begin{aligned}
x^* = \frac{x}{L} , \quad y^* = \frac{y}{L} , \quad u^* = \frac{u}{U} , \quad v^* = \frac{v}{V} \\
P^* = \frac{P}{\rho U^2} , \quad \alpha_2^* = \frac{2}{\rho L^2} , \quad \alpha_3^* = \frac{3}{\rho L^2}
\end{aligned}
\right\}
\tag{8}
$$

where L is some characteristic length and U is a characteristic velocity. For simplicity we choose **immediately** to drop the * notation although it is still implied.

Secondly we introduce a stream function $\psi(x,y)$ such that

$$
u = \frac{\partial \psi}{\partial y} \quad \text{and} \quad v = - \frac{\partial \psi}{\partial x} . \tag{9}
$$

With this definition the equation of continuity (3) is satisfied identically.

Finally we substitute the expressions (5)-(7) for the stress components P'_{ik} into equations (1) and (2) and eliminate the pressure. The final equations which we then obtain are

$$\frac{\partial\psi}{\partial y}\frac{\partial\zeta}{\partial x} - \frac{\partial\psi}{\partial x}\frac{\partial\zeta}{\partial y} = \frac{1}{Re}\nabla^2\zeta + \alpha_2\left[\frac{\partial\psi}{\partial y}\frac{\partial}{\partial x} - \frac{\partial\psi}{\partial x}\frac{\partial}{\partial y}\right]\nabla^2\zeta \tag{10}$$

where ζ is the vorticity given by

$$\zeta = \nabla^2\psi\ , \tag{11}$$

∇^2 is the usual Laplacian operator

$$\nabla^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\ , \tag{12}$$

and Re is a Reynolds number given by

$$Re = \rho\frac{UL}{\alpha_1}\ . \tag{13}$$

For L we choose the height h of the channel and for U we take the maximum value of the inlet velocity profile, i.e. $U = \bar{P}\,h^2/4\alpha_1$ where $\bar{P}$ is the pressure gradient producing the flow. Our Reynolds number becomes

$$Re = \rho\frac{\bar{P}\,h^3}{4\,\alpha_1^2} \tag{14}$$

Simplified Analysis of Tanner and Pipkin [5]

If one assumes that the flow is so slow that one may neglect inertia, then the disturbance due to the slot is symmetrical about the slot centerline, and therefore the pressure P_0 in a Newtonian liquid is constant along this line. If one further assumes that the slot is sufficiently deep that there is negligible motion at the bottom of it then this constant value is the pressure P_H at the bottom of the slot. Under these restrictions Tanner and Pipkin show that for a second order fluid the stress along the centerline is given by

$$P_{ik} = -P_H \delta_{ik} + 2\alpha_1 e_{ik}^{(1)} + 4\alpha_2[e_{im}^{(1)} e_k^{(1)m} - \frac{1}{2}(\mathrm{trace}(e_{im}^{(1)} e_k^{(1)m})\delta_{ik})] \tag{15}$$
$$+ \alpha_3[2e_{ik}^{(2)} - (\frac{1}{\alpha_1} \frac{DP_0}{DT} - 3\ \mathrm{trace}\ (e_{im}^{(1)} e_k^{(1)m}))\delta_{ik}]$$

If one now further assumes that the slot in the bottom plate is so narrow (h/b large) that the flow near the top plate is negligibly disturbed then it is fairly straightforward to show that the centerline thrust P_w on the top plate is given by

$$P_w = P_H - \frac{1}{2}\alpha_2(\frac{\partial u}{\partial y})^2 . \tag{16}$$

Since the first normal stress difference N_1 is given by

$$N_1 = P_{xx} - P_{yy} = -2\alpha_2(\frac{\partial u}{\partial y})^2 \tag{17}$$

then from (16) and (17) we have the simple relationship between the first normal stress difference and the hole pressure $P_w - P_H$ namely

$$P_w - P_H = \frac{1}{4} N_1 . \tag{18}$$

This result suggests that measurement of the hole pressure is likely to give a direct measure of the elasticity of a liquid. This analysis has been extended by Kearsley [7] who was able to show that for rectilinear flows along a slot the hole pressure is one half the second normal stress difference. Higashitani and Pritchard [6] using a somewhat different approach have confirmed the Tanner-Pipkin result although here again the methods used depend crucially on the flow patterns being symmetric about the centerline of the slot. This is a severe restriction. In practical instruments one has a finite Reynolds number and one must expect inertial effects to produce asymmetrical streamlines within the slot. To investigate fully the relationship between hole-pressure and the first normal stress difference one must retain the inertia terms in Equation (11) and derive solutions for some finite positive Reynolds number. In this case, however, analytical solutions are out of the question and one must turn to numerical techniques.

Numerical Solution

In order to solve the equations numerically it is convenient to introduce an intermediate variable ξ as follows. We write Equation (10) in the form

$$\frac{1}{Re} \nabla^2 \zeta - \frac{\partial\psi}{\partial y}\frac{\partial\zeta}{\partial x} - \frac{\partial\psi}{\partial x}\frac{\partial\zeta}{\partial y} = \xi \tag{19}$$

and

$$\alpha_2 \left[\frac{\partial\psi}{\partial y}\frac{\partial\xi}{\partial x} - \frac{\partial\psi}{\partial x}\frac{\partial\xi}{\partial y}\right] + \frac{\xi}{Re} = -\alpha_2\left[\frac{\partial\psi}{\partial y}\frac{\partial}{\partial x}\left(\frac{\partial\psi}{\partial y}\frac{\partial\zeta}{\partial x} - \frac{\partial\psi}{\partial x}\frac{\partial\zeta}{\partial y}\right) - \frac{\partial\psi}{\partial x}\frac{\partial}{\partial y}\left(\frac{\partial\psi}{\partial y}\frac{\partial\zeta}{\partial x} - \frac{\partial\psi}{\partial x}\frac{\partial\zeta}{\partial y}\right)\right] \tag{20}$$

$\xi(x,y)$ represents a deviation from Newtonian behaviour. If α_2 is zero then, from (20), ξ is identically zero and (11) and (19) reduce to the Navier-stokes equations.

To solve these equations together with (11), we use finite difference techniques and introduce a mesh (x_i, y_i) (see Figure 2) defined by

$$\left.\begin{array}{ll} x_i = i\,\Delta x\,, & i = 0,1,\ldots,M \\ y_i = j\,\Delta y\,, & j = 0,1,\ldots,N \end{array}\right\} . \tag{21}$$

To discretize the Laplacian operators in (11) and (19) we use standard five point difference formulae

$$\nabla^2 A_{ij} \equiv \frac{A_{i+1j} - 2A_{ij} + A_{i-1j}}{(\Delta x)^2} + \frac{A_{ij+1} - 2A_{ij} + A_{ij-1}}{(\Delta y)^2} \tag{22}$$

For the non-linear terms in (19) it is necessary to take special measures to insure diagonal dominance of the difference equations. Otherwise it has been found that the iterative scheme to be applied to solve these equations will not converge for Reynolds numbers in excess of about unity. One technique used by Schafenacher [14] is to introduce additional terms into the difference approximation for the x derivative of ζ, based on the previous iterative step. Another more popular technique is to make use of so-called 'upwind' differencing. In this case, one-sided difference approximations to the first derivatives of ζ are used, the particular approximation i.e. forward or backward, being chosen according to the signs

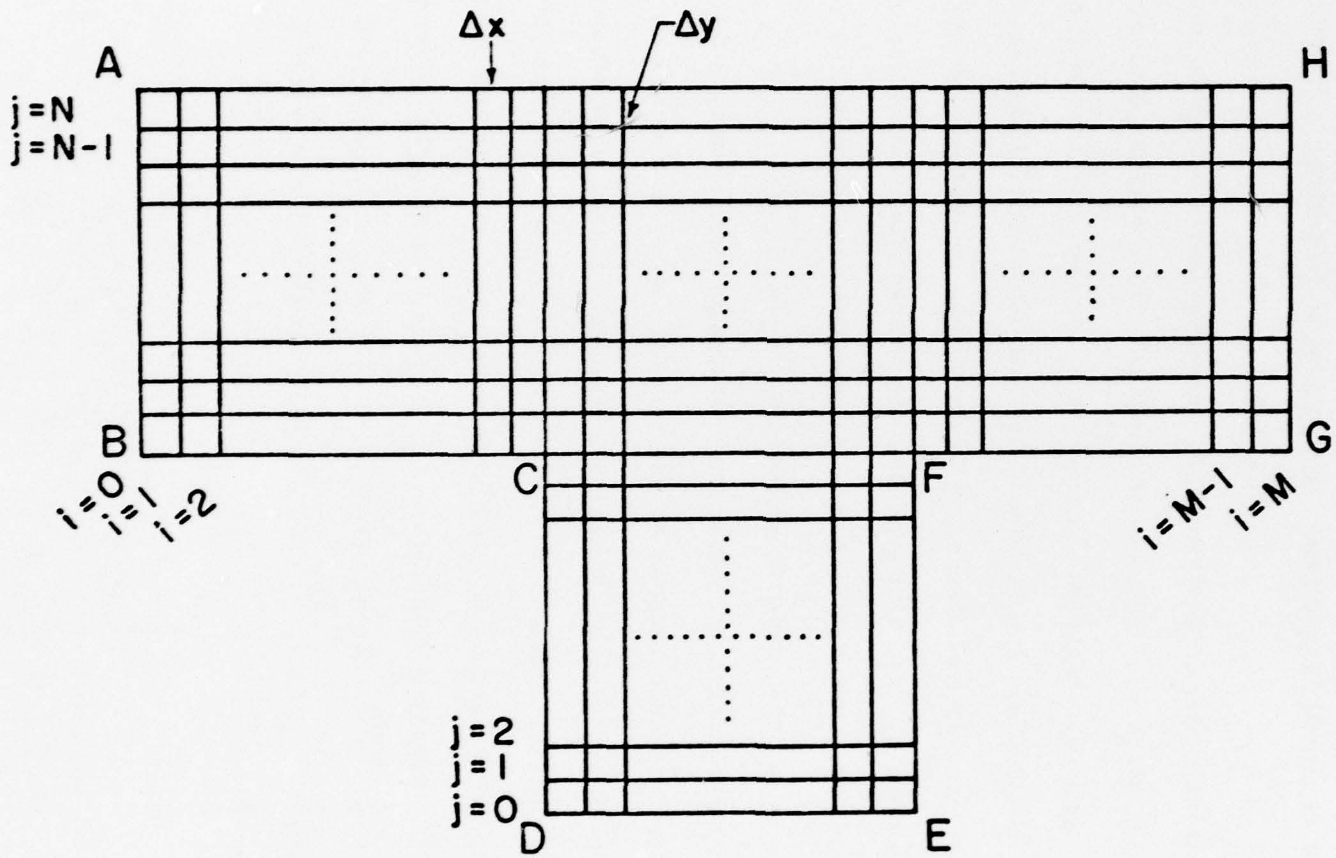

Figure 2

The finite difference grid

of the corresponding derivatives of ψ , to ensure diagonal dominance. In the work considered here, the latter technique has proved to be a somewhat more satisfactory method although one has to accept a loss of accuracy due to the first order nature of the one-sided difference approximations. Greater accuracy can be recovered however, if required, by applying a difference correction procedure (See, for example, Dennis and Cheng [19]).

Discretization of Equation (20) presents similar diagonal dominance problems and we must again use one-sided differences for the first derivatives of ξ .

For boundary conditions we have that ψ and its first derivatives are zero on solid boundaries, and for the inlet and outlet conditions we impose Poiseuille flow profiles. As an alternative condition at the outlet, ψ may be assumed to be independent of x , although this slows down the convergence of the iterative procedures. For the range of Reynolds numbers considered here, it proves adequate to impose both inlet and outlet profiles.

The three difference equations may now be solved using successive over-relaxation, although because of the coupled nature of the system it is necessary to set up an inner/outer iteration procedure. Normally to start such a procedure one makes initial guesses $\psi^{(0)}, \zeta^{(0)}$ and $\xi^{(0)}$ for the interior of the finite difference mesh, and then using an appropriate Taylor Series formula one constructs from $\psi^{(0)}$ boundary values for $\zeta^{(0)}$. (See, for example, Greenspan and Schultz [20]). From $\zeta^{(0)}$ one may then solve (11) to give a new approximation $\psi^{(1)}$ for ψ . Using $\psi^{(1)}$ and $\xi^{(0)}$ one then proceeds to solve (19) for $\zeta^{(1)}$, and so on. The sequence

$$\begin{tikzcd}
\psi^{(0)} \arrow[d] & \psi^{(1)} \arrow[d] \arrow[dd, bend left] & \psi^{(2)} \arrow[d] \arrow[dd, bend left] & \cdots\cdots \\
\zeta^{(0)} \arrow[ur] & \zeta^{(1)} \arrow[ur] \arrow[d] & \zeta^{(2)} \arrow[d] & \cdots\cdots \\
\xi^{(0)} \arrow[uur] \arrow[ur] & \xi^{(1)} \arrow[uur] \arrow[ur] & \xi^{(2)} & \cdots\cdots
\end{tikzcd} \tag{23}$$

of approximations (outer iterates) to each of the three variables is calculated until the nth and $(n+1)$st outer iterate agree to some predetermined tolerance. In some cases convergence problems are experienced and it is found necessary to apply smoothing between each outer iterate. When, for example, a new approximation $\psi^{(n+1)}$ is computed, then a weighted mean $\bar{\psi}^{(n+1)}$ of this value and the previous value given by

$$\bar{\psi}^{(n+1)} = \rho_2\, \psi^{(n)} + (1 - \rho_s)\, \psi^{(n+1)} , \quad 0 \leq \rho \leq 1 \tag{24}$$

is actually used for ψ for the next stage of the outer iteration procedure. A similar smoothing is applied to ζ and to ξ.

Greenspan and Shultz [20] have suggested that it may be possible to avoid this inner/outer iteration procedure. They solve a number of problems for the Navier Stokes equations in which successive over-relaxation is applied simultaneously to the vorticity and stream function equations. They found that this was a much faster procedure which has the added advantage of not requiring smoothing parameters. However, attempts to implement these ideas for the problems considered here proved only to be partly successful. Simultaneous iteration of all three equations (11), (19) and (20) did not produce a convergent solution, and even if only the first two equations were iterated together then a convergent solution is only obtained if a certain degree of smoothing is applied. In this case, a smoothing correction is applied at the end of each complete sweep through the mesh of the ψ and ζ iterations. This method did, however, prove to be much faster than an inner/outer iteration scheme. ψ and ζ are now treated as a pair and combined with ξ in a normal inner/outer iteration as follows

$$\begin{tikzcd} \psi^{(0)},\zeta^{(0)} & \psi^{(1)},\zeta^{(1)} \arrow[d] & \psi^{(2)},\zeta^{(2)} \arrow[d] \\ \xi^{(0)} \arrow[ur] & \xi^{(1)} \arrow[ur] & \xi^{(2)} \end{tikzcd} \tag{25}$$

In spite of the efforts set out above, a convergent solution proved difficult to obtain for Reynolds numbers in excess of about 50. With even greater smoothing and the use of a finer finite difference mesh it is likely that the range of Reynolds numbers could be extended. For our purposes, we are most interested in somewhat slower flows, and therefore no attempt was made to extend the analysis to very high Reynolds numbers.

The Hole Pressure

It is difficult to define precisely what we mean by the hole pressure. Essentially if a pressure transducer is flush mounted in the bottom wall of the slot and gives a reading P_H, and a second transducer mounted directly opposite in the top plate gives a reading P_w then the difference P_w-P_H is the hole pressure.

However difficulty arises as to how to simulate the action of the transducer diaphragm. It was decided to compute an average force exerted by the fluid on some portion of the bottom wall of the slot together with the equivalent average force exerted on a similar portion of the top plate. The range of integration chosen in the averaging was the complete bottom wall of the slot except for a portion Δx at each end. This avoids difficulties at the corners C and F.

The hole pressure ΔP is then given by

$$\Delta P = \int (P_{yy_{\text{top plate}}} - P_{yy_{\text{slot}}})\,dx/(b-2\Delta x) \tag{26}$$

where P_{yy} is the normal component of the stress tensor given by

$$P_{yy} = -P + P'_{yy} \quad . \tag{27}$$

The difference in the values of the fluid pressure P is given by

$$P_{\text{top plate}} - P_{\text{slot}} = \int_{\text{slot}}^{\text{top plate}} \frac{\partial P}{\partial y}\,dy$$

where from (2), (6) and (7)

$$
\begin{aligned}
\frac{\partial P}{\partial y} = \frac{1}{Re}\left[\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right] - \left[u\,\frac{\partial v}{\partial x} + v\,\frac{\partial v}{\partial y}\right] \\
+ 2\alpha_2 \Bigg[\frac{1}{2}\,\frac{\partial u}{\partial x}\Big(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 u}{\partial x \partial y} + \frac{u}{2}\Big(\frac{\partial^3 v}{\partial x^3}\;\frac{\partial^3 u}{\partial x^2 \partial y}\Big) + \frac{1}{2}\,\frac{\partial v}{\partial x}\Big(\frac{\partial^2 v}{\partial x \partial y} + \frac{\partial^2 u}{\partial y^2}\Big) \\
+ \frac{v}{2}\Big(\frac{\partial^3 v}{\partial x^2 \partial y} + \frac{\partial^3 u}{\partial x \partial y^2}\Big) + \frac{\partial^2 v}{\partial x^2}\,\frac{\partial v}{\partial y} + \frac{\partial v}{\partial x}\,\frac{\partial^2 v}{\partial x \partial y} + \frac{\partial^2 u}{\partial x \partial y}\,\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\,\frac{\partial^2 u}{\partial x^2} \\
+ \frac{\partial u}{\partial y}\,\frac{\partial^2 v}{\partial x \partial y} + u\,\frac{\partial^3 v}{\partial x \partial y^2} + \frac{\partial v}{\partial y}\,\frac{\partial^2 v}{\partial y^2} + v\,\frac{\partial^3 v}{\partial y^3} + \frac{\partial^2 u}{\partial y^2}\Big(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\Big) \qquad (28) \\
+ \frac{\partial u}{\partial y}\Big(\frac{\partial^2 v}{\partial x \partial y} + \frac{\partial^2 u}{\partial y^2}\Big) + 4\Big(\frac{\partial v}{\partial y}\Big)\Big(\frac{\partial^2 v}{\partial y^2}\Big)\Bigg] \\
+ 4\alpha_3\left[2\,\frac{\partial v}{\partial y}\,\frac{\partial^2 v}{\partial y^2} + \frac{1}{2}\Big(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\Big)\Big(\frac{\partial^2 v}{\partial x \partial y} + \frac{\partial^2 u}{\partial y^2}\Big)\right] .
\end{aligned}
$$

Both integrations indicated above are carried out numerically using a simple Simpson's rule algorithm.

In order to compare our result with the Tanner-Pipkin result, a final ratio R, corrected for inertial effects, is computed, where

$$R = (\Delta P_{\text{second order fluid}} - \Delta P_{\text{Newtonian}})/N_1 \qquad (29)$$

and where N_1 is the first normal stress difference defined as

$$N_1 = P_{xx} - P_{yy} \; . \qquad (30)$$

To compute N_1 we take average values of the stress components over the region of the top plate opposite the slot.

For the different slot geometries considered, different finite difference meshes were constructed. In all cases, however, approximately 20 grid cells were chosen to represent the y variation of solution in the main portions of the channel away from the slot, and likewise approximately 20 grid cells were chosen to represent the x variation within the slot. The over-all total of grid points was dependent on the particular geometry under consideration. All computation was carried out on a UNIVAC 1110 computer.

Results

Streamlines for various flow conditions are shown in Figures 3-8. It will be noted that in all cases a secondary flow, very much weaker than the flow in the mainstream, is set up in the slot.

For creeping flow it is well known that the velocity field of a Newtonian liquid is identical with that of a second order fluid. See for example Giesekus [21], Tanner [22]. It is of interest to investigate just how much the two flow fields differ when inertial effects are included. Since the first normal stress difference acts like a tension along the streamlines that tends to pull the liquid out of the slot one would expect that part of the Newtonian flow field which dips into the slot to straighten for a second order fluid. Certainly for the range of Reynolds numbers considered here any changes due to elasticity proved to be in the expected direction but these changes are extremely small in magnitude. This observation was also made by Crochet and Bezy [17]. In Figure 3 we have plotted streamlines for a square slot of depth equal to the main channel height. The full lines are for a Newtonian liquid at a Reynolds number of 25 and the dashed lines indicate changes due to elasticity when $\alpha_2^* = -.1$. It can be seen just how little the flow field is modified by elasticity even though inertial effects are sufficiently large to cause the overall flow pattern to deviate quite considerably from the pattern, symmetric about the centerline, which one obtains for creeping flow.

In Figures 4 and 5 we can see how the gradual influence of inertia is felt as the Reynolds number increases. At a very small Reynolds number, the streamlines across the whole channel are modified by the presence of the slot, those close to the slot being drawn quite deeply into the slot. As the Reynolds number increases the streamlines straighten until only those quite close to the slot are affected. Another noticable effect of increasing the speed of the mainstream is to strengthen the secondary flow in the slot although it always remains several orders of magnitude weaker than the main flow.

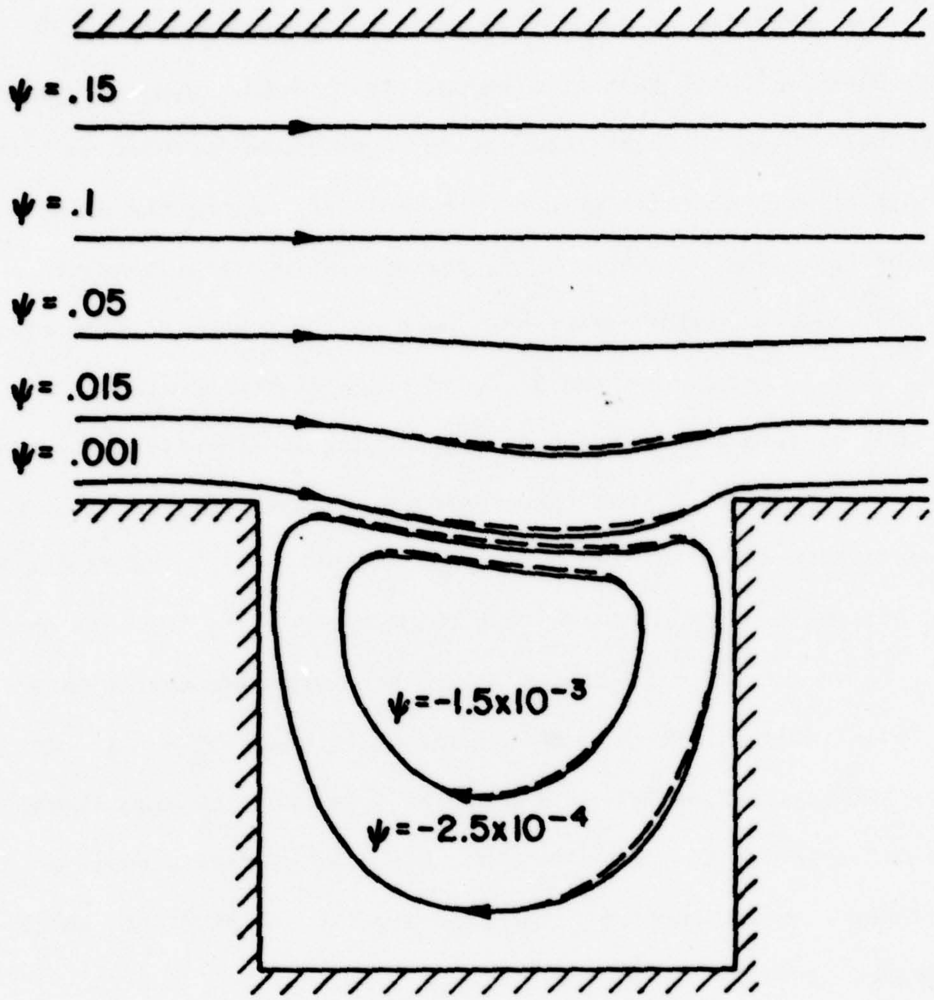

Figure 3

Streamline projections for a square slot with $h/b = 1$. Re = 25

Full lines - Newtonian

Broken lines - second order fluid ($\alpha_2^* = -.1$)

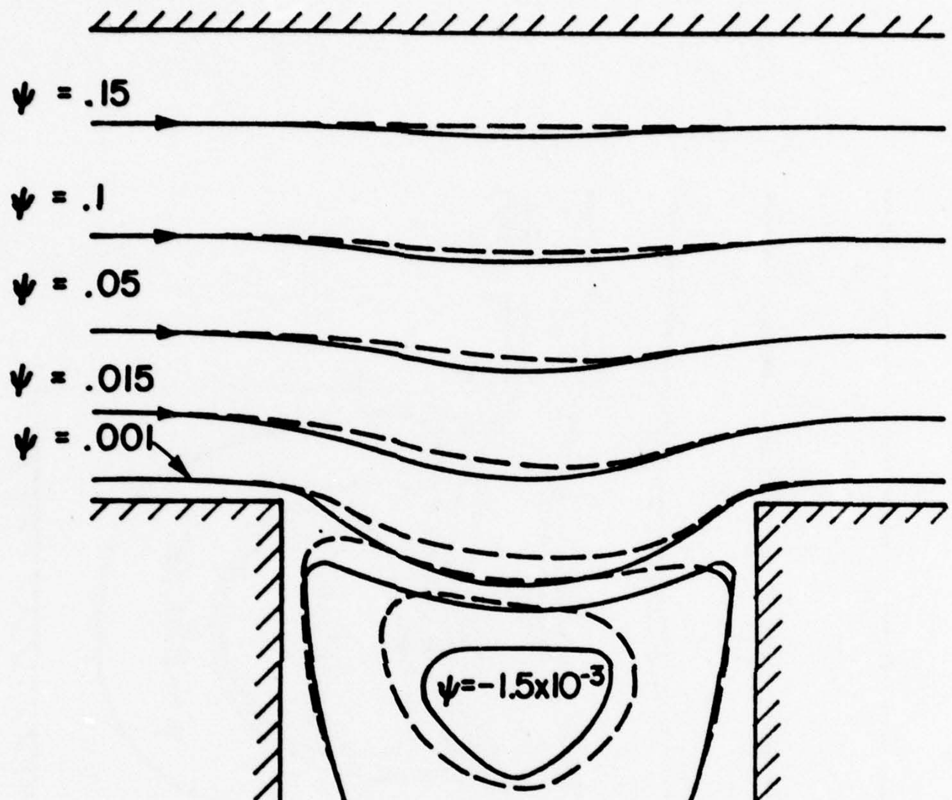

Figure 4

Streamline projections for a square slot with $h/b = 1$ and for a second order fluid ($\alpha_2^* = -.1$)

Full lines - Re = 1. Broken lines - Re = 10.

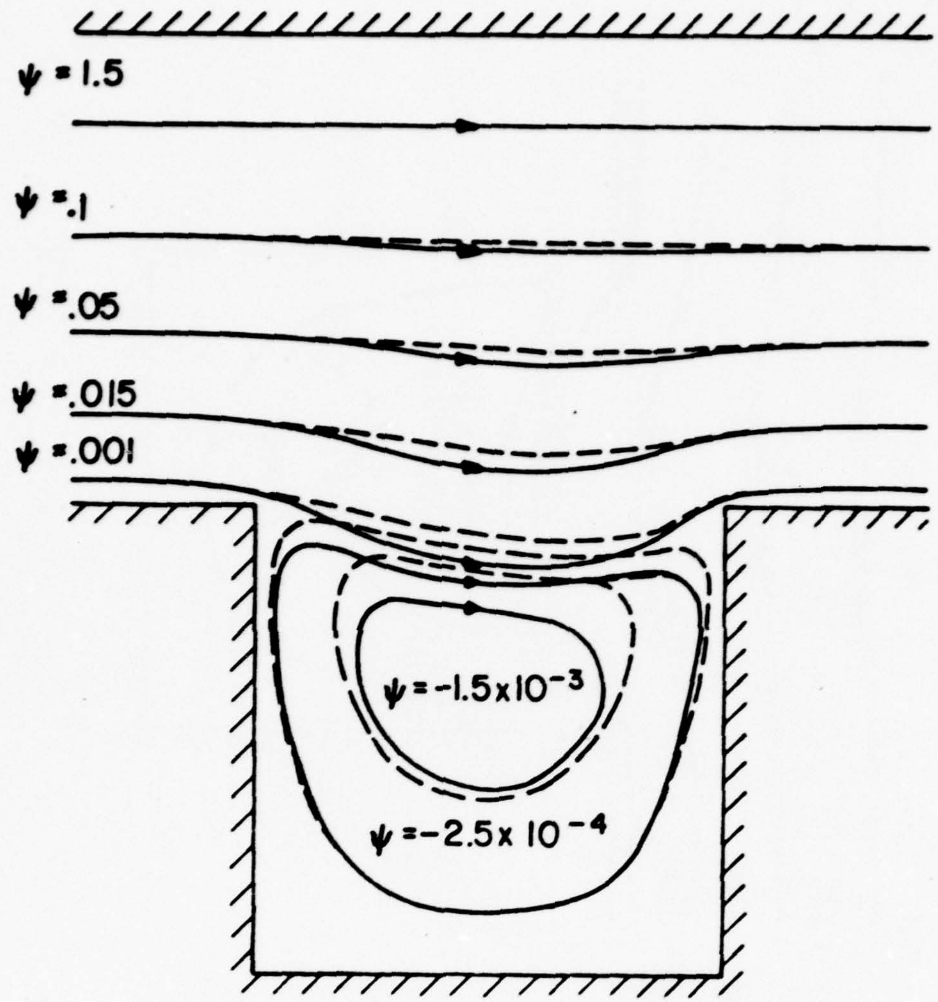

Figure 5

Streamline projections for a square slot with $h/b = 1$ and for a second order fluid $(\alpha_2^* = -.1)$

Full lines - Re = 10. Broken lines - Re = 25

In Figures 6-8 we have considered how the flow field is affected by changes in the channel/slot geometry. Two factors are relevant here, namely the slot depth and the slot width. Figures 6 and 7 show streamlines plotted for slots of half and double depth respectively. A comparison of Figures 4, 6 and 7 shows that the main flow is relatively unaffected by slot depth and that in all cases the fluid is virtually stationary at the bottom of the slot. One might expect therefore that hole pressures would be relatively insensitive to slot depth and we shall investigate this point later. One consequence of changing the slot depth, however, is a considerable change in the secondary flow, and, in particular, for slots of double depth, one sees the onset of a second separate circulatory motion in the slot.

Figure 8 shows streamlines for a slot of half width. Intuitively one might expect inertial effects to be reduced for a narrower slot and the results seem to confirm this, for the streamlines in the main channel are straighter compared with a square slot, the overall pattern is more symmetric about the slot center-line, and very little change is seen when the Reynolds number is increased from 10 to 25.

If we turn our attention now to the hole pressure, Figures 9 and 10 show plots of the hole pressure as a function of distance across the slot. Further tabulated values for representative points across the slot are calculated in Table 1. It can be seen from the figures that for the Newtonian case (dashed lines), a positive value in the first part of the slot is to some extent cancelled by a negative value in the remainder of the slot so that the net integrated hole pressure recorded by two transducers will be very small. This is not the case for the elastic liquid (full lines) where the curves remain positive throughout the slot. One would expect, therefore, the hole pressure contribution due to elasticity to be at least an order of magnitude greater than that due to inertial effects at the Reynolds numbers considered here.

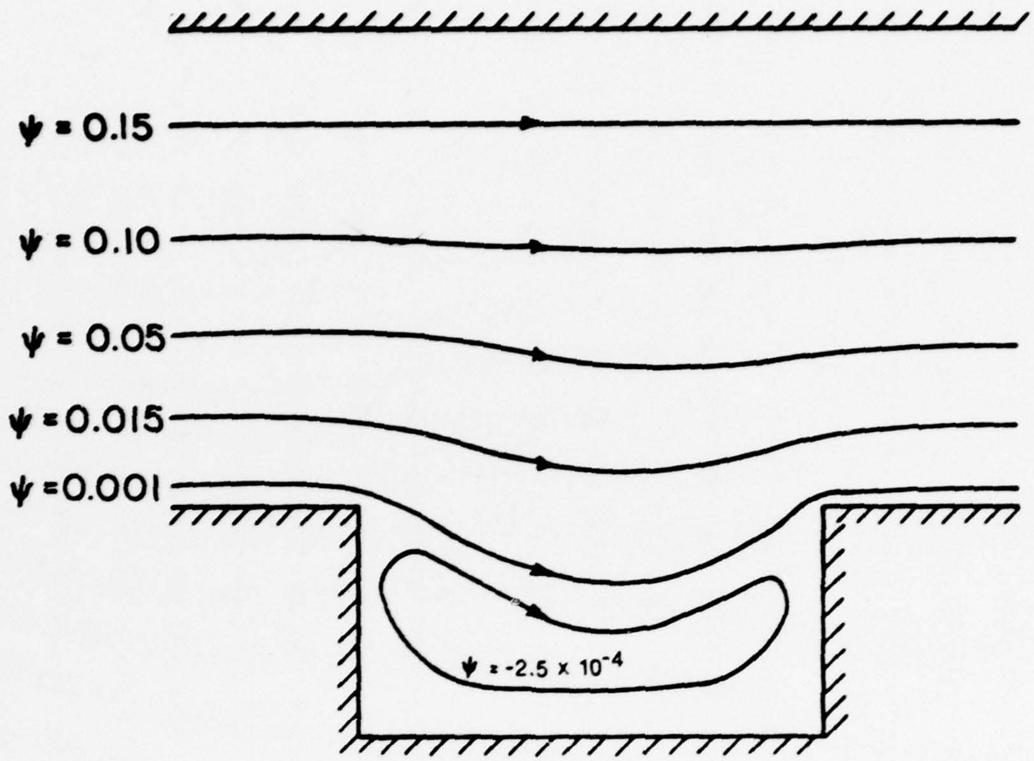

Figure 6

Streamline projections for a second order fluid $(\alpha_2^* = -.1)$ in a shallow slot with $h/d = 2$. $Re = 10$.

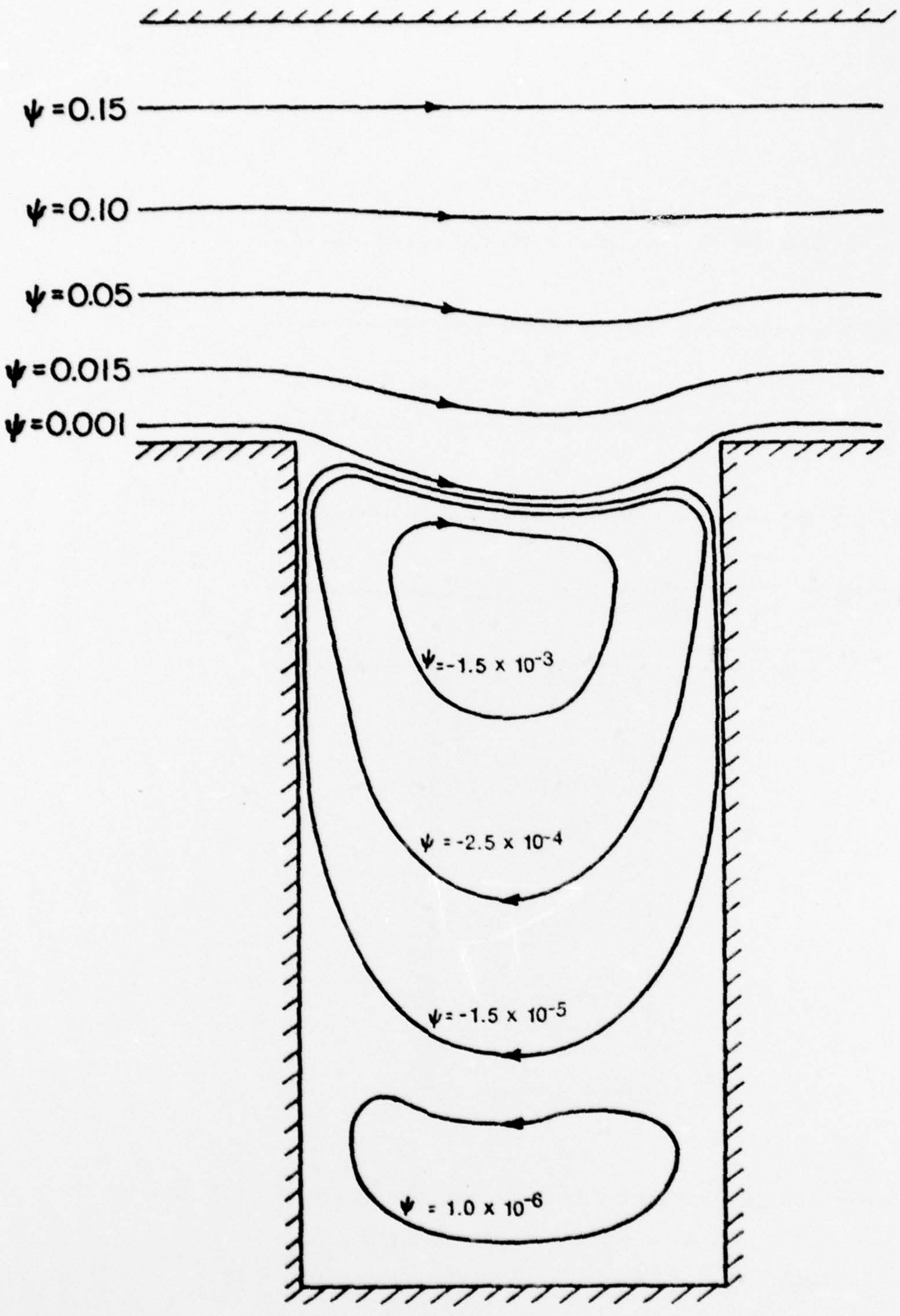

Figure 7

Streamline projections for a second order fluid ($\alpha_2^* = -.1$) in a deep slot with $h/d = 0.5$. $Re = 10$.

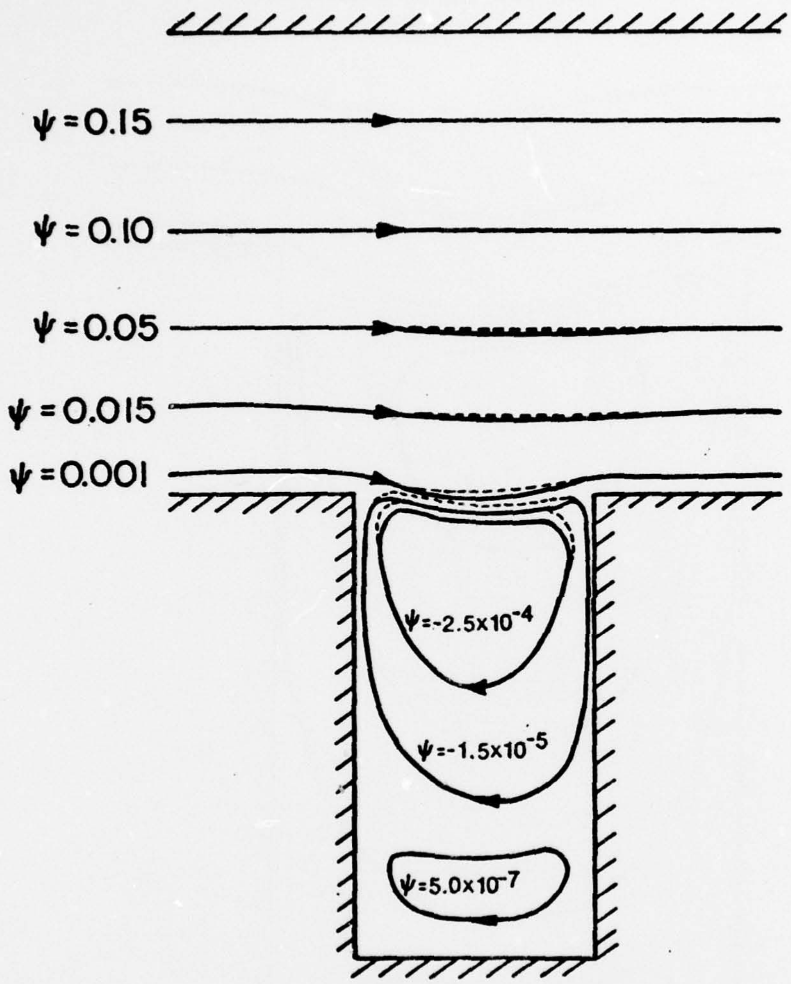

Figure 8

Streamline projections for a second order fluid $(\alpha_2^* = -.1)$

in a narrow slot $h/b = 2$.

Full lines - Re = 10. Broken lines - Re = 25.

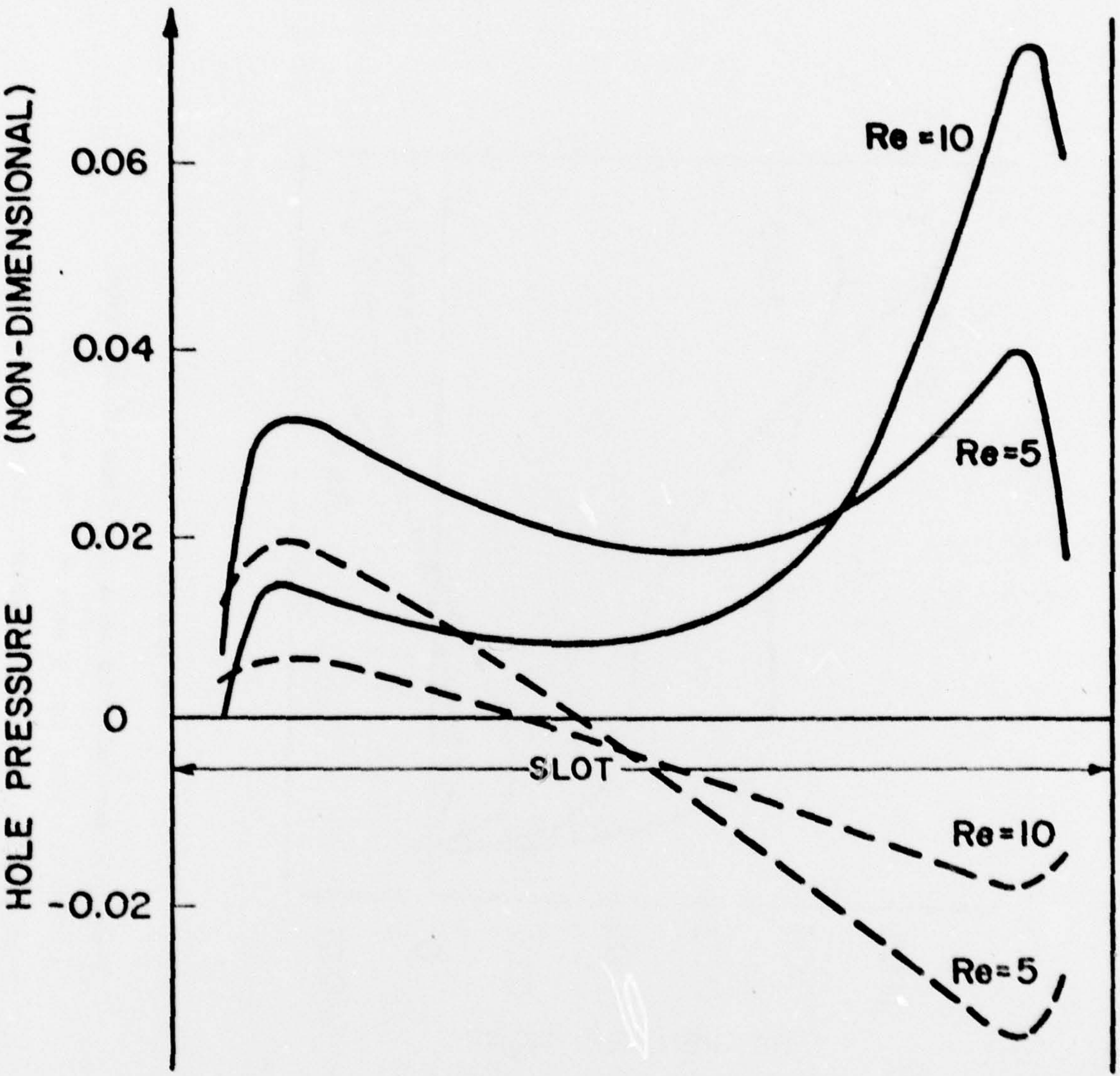

Figure 9

Hole pressure plotted as a function of distance across the slot for a square slot with h/b = 1.

Full lines - $\alpha_2^* = -.1$. Broken lines - Newtonian.

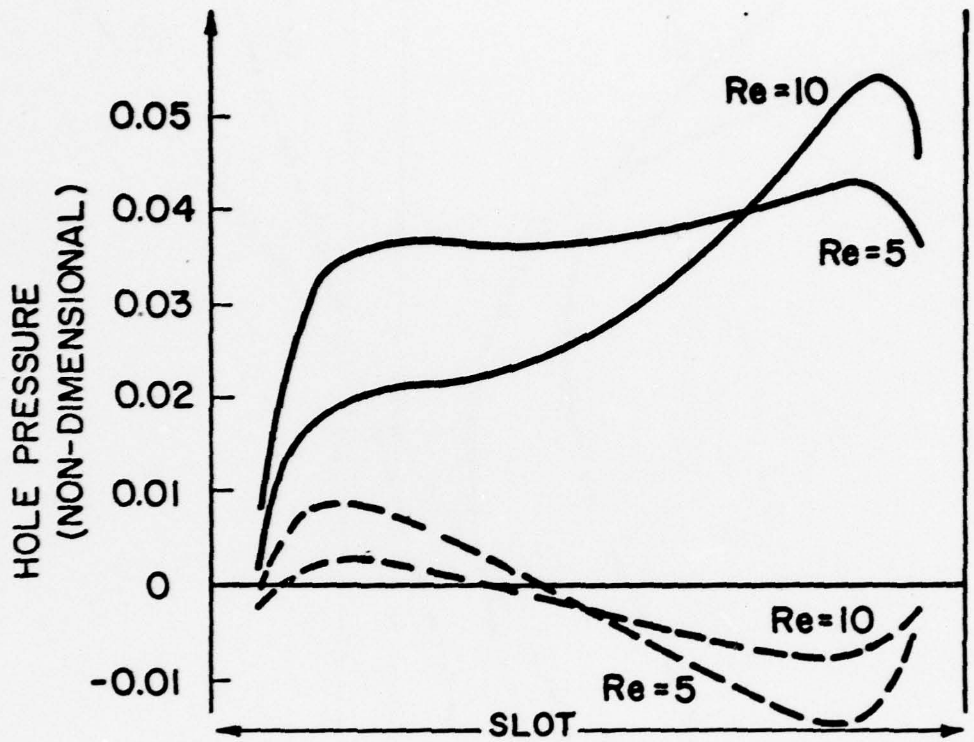

Figure 10

Hole pressure plotted as a function of distance across the slot for a narrow slot with h/b = 2. Full lines - $\alpha_2^* = -.1$. Broken lines - Newtonian.

x	SQUARE SLOT h/b = 1								NARROW SLOT h/b = 2			
	Newtonian				Second Order Fluid ($\alpha_2^* = -.1$)				Newtonian		$\alpha_2^* = -.1$	
	Re=.1	Re=5	Re=10	Re=20	Re=.1	Re=5	Re=10	Re=20	Re=5	Re=10	Re=5	Re=10
.1	5.66	1.94×10^{-2}	6.74×10^{-3}	1.46×10^{-3}	5.83	3.19×10^{-2}	1.47×10^{-2}	8.26×10^{-3}	5.77×10^{-3}	7.52×10^{-4}	2.14×10^{-3}	1.08×10^{-2}
.2	4.50	1.51×10^{-2}	5.04×10^{-3}	1.08×10^{-3}	4.65	2.92×10^{-2}	1.20×10^{-2}	5.90×10^{-3}	8.35×10^{-3}	2.71×10^{-3}	3.57×10^{-2}	2.02×10^{-2}
.3	2.97	8.82×10^{-3}	2.35×10^{-3}	1.51×10^{-4}	3.10	2.43×10^{-2}	9.27×10^{-3}	3.90×10^{-3}	5.40×10^{-3}	1.35×10^{-3}	3.64×10^{-2}	2.16×10^{-2}
.4	1.47	2.27×10^{-3}	-5.40×10^{-4}	-9.85×10^{-4}	1.57	2.08×10^{-2}	8.16×10^{-3}	2.75×10^{-3}	1.51×10^{-3}	-4.55×10^{-4}	3.61×10^{-2}	2.33×10^{-2}
.5	6.05×10^{-3}	-4.51×10^{-3}	-3.65×10^{-3}	-2.35×10^{-3}	6.17×10^{-2}	1.88×10^{-2}	8.83×10^{-3}	2.57×10^{-3}	-2.59×10^{-3}	-2.36×10^{-3}	3.64×10^{-2}	2.66×10^{-2}
.6	1.48	-1.16×10^{-2}	-7.01×10^{-3}	-3.97×10^{-3}	-1.44	1.88×10^{-2}	1.24×10^{-2}	4.55×10^{-3}	-6.78×10^{-3}	-4.3×10^{-3}	3.77×10^{-2}	3.21×10^{-2}
.7	-2.99	-1.92×10^{-2}	-1.07×10^{-2}	-5.82×10^{-3}	-2.97	2.21×10^{-2}	2.16×10^{-2}	1.21×10^{-2}	-1.09×10^{-2}	-6.23×10^{-3}	3.98×10^{-2}	3.96×10^{-2}
.8	-4.51	-2.73×10^{-2}	-1.47×10^{-2}	-7.80×10^{-3}	-4.51	2.97×10^{-2}	4.07×10^{-2}	3.34×10^{-2}	-1.43×10^{-2}	-7.75×10^{-3}	4.19×10^{-2}	4.87×10^{-2}
.9	-5.68	-3.41×10^{-2}	-1.81×10^{-2}	-9.24×10^{-3}	-5.70	3.96×10^{-2}	7.21×10^{-2}	8.20×10^{-2}	-1.03×10^{-2}	-5.68×10^{-3}	4.00×10^{-2}	5.08×10^{-2}

Table 1

Hole pressures tabulated against non-dimensional distance across the slot for various geometries and Reynolds numbers

Although the parameter α_3 does not appear in the equations for the streamlines, it does affect pressure calculations. Throughout this work a variation of α_3 within an acceptable range of values proved to have a virtually negligible effect on the results.

In Figure 11 we have plotted the ratio R as a function of Reynolds number. It is clear that the assumptions made in the Tanner-Pipkin theory breakdown fairly rapidly as the Reynolds number increases above unity although there is still a very close relationship between the hole pressure and the first normal stress difference. Also plotted in Figure 11 are some of the results of Crochet and Brezy. It should be said that it is not clear from their work whether they have applied an inertial correction as we have, by subtracting off the Newtonian contribution to the hole-pressure. (See Equation (29)). This correction would bring their results a little closer to those computed here although some difference is still apparent.

Finally we return to the question of geometrical effects in hole pressure measurement. In Table 2 the ratio R is tabulated for four different slot geometries. As suggested earlier, it would seem that the relationship between hole pressure error and the first normal stress difference is relatively insensitive to changes in the depth of the slot. The width of the slot is a more important factor however. A narrower slot leads to less inertial distortion of the flow field and, as a result, a value of R closer to that predicted by the Tanner-Pipkin theory.

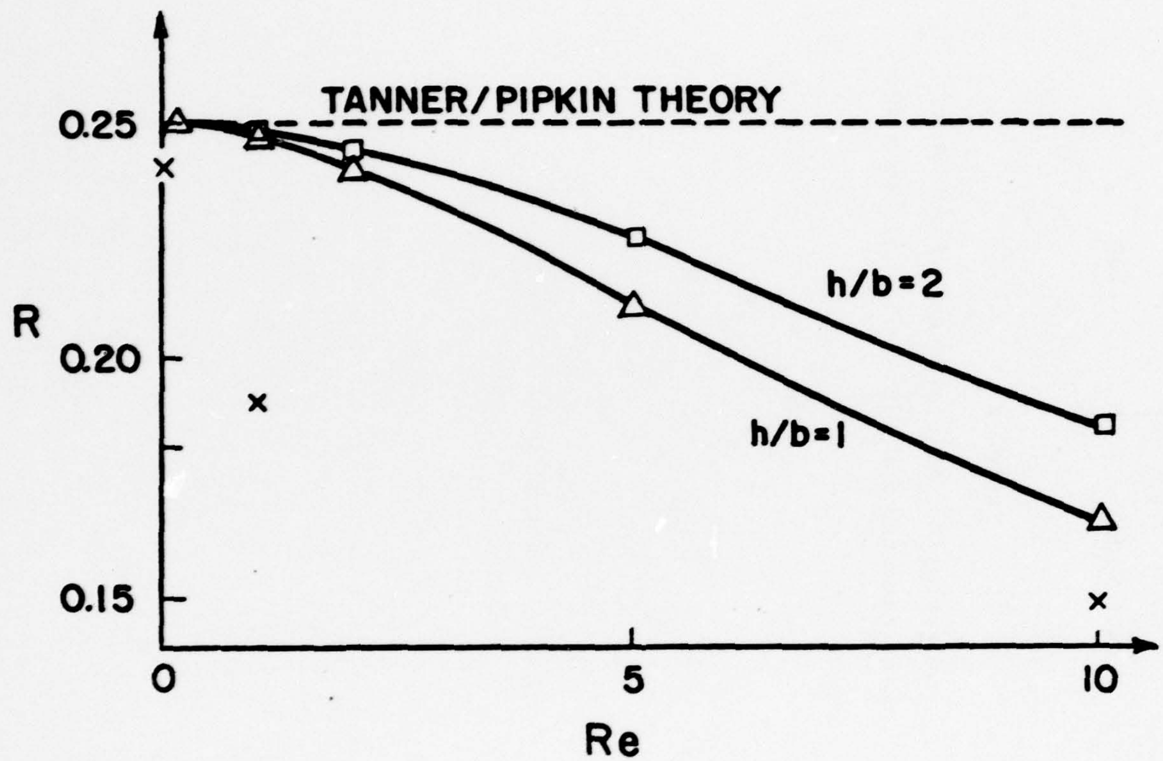

Figure 11

The ratio R of hole pressure to first normal stress difference plotted as a function of Reynolds number.

Δ - square slot with h/b = 1.

□ - narrow slot with h/b = 2.

x - results due to Crochet & Bezy (16).

Type of Slot	R
Square	.166
Half depth	.167
Double depth	.166
Half width	.186

Table 2

The ratio R tablulated as a function of slot geometry for $\alpha_2^* = -.1$ and Re = 10.

Conclusions

The present investigation into the determination of the first normal stress difference of a visco-elastic liquid by the measurement of hole-pressures indicates that the problem is somewhat more complex than that modelled by the Tanner/Pipkin theory. Although elasticity is found to have very little influence on streamline patterns, even at quite appreciable Reynolds numbers, inertial distortion of the flow field results in considerable change in the relationship between the hole-pressure and the first normal stress difference when $Re > 1$. However, our results confirm the Tanner/Pipkin theory for Reynolds numbers below unity.

Slot geometry is not found to be an important factor as far as slot depth is concerned, but narrower slots are found to reduce inertial distortion of the flow.

The results confirm that elastic and inertial contributions to the hole pressure are additive in the sense that $\Delta P_{elastic} - \Delta P_{Newtonian}$ is proportional to N_1 for given Re and h/b.

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Liquid-filled holes used for pressure measurements of visco-elastic liquids give rise to systematic hole pressure 'errors'. Tanner and Pipkin have presented analysis for flows of a second order fluid in which they derive a simple relation between the first normal stress difference and the hole pressure for flow situations where Reynolds numbers are very small. Implicit in the analysis is the assumption that the streamlines are symmetric about the hole center line. In this paper, using a numerical solution, we investigate the relationship between the hole pressure and the first normal stress difference for a range of → over

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Abstract (continued)

Reynolds numbers where inertial effects are not negligible. The ratio of hole pressure/first normal stress difference is found to vary from 0.25 to 0.16 as the Reynolds number is varied from 1 to 10. Streamline patterns are presented for Poiseuille flow of a second order fluid over a slot cut into one wall of an otherwise straight channel. Various geometries are considered. The results naturally include those for an incompressible Newtonian liquid at non-zero Reynolds numbers.

† It has been customary in the past to apply the term hole pressure 'error' to the quantity $P_H - P_w$ where P_H is the pressure measured in the static liquid at the end of a hole in a channel wall and P_w is the pressure which would be exerted on the wall by liquid flowing in the channel if the hole were not present. The basic aim of this paper is to relate this 'quantity' to material properties of the liquid, and as such the term 'error' is misleading since the 'quantity' will itself be subject to error. We choose, therefore, to refer to the 'hole pressure' defined as $P_w - P_H$. It will be noted that the sign of the hole pressure is chosen to be positive for elastic liquids.